# An explicit representation of Verblunsky coefficients

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**Abstract**

We prove a representation of the partial autocorrelation function (PACF) of a stationary process, or of the Verblunsky coefficients of its normalized spectral measure, in terms of the Fourier coefficients of the phase function. It is not of fractional form, whence simpler than the existing one obtained by the second author. We apply it to show a general estimate on the Verblunsky coefficients for short-memory processes as well as the precise asymptotic behaviour, with remainder term, of those for FARIMA processes.

*Keywords:* Verblunsky coefficients, Partial autocorrelation functions, Phase functions, FARIMA processes, Long memory
*2000 MSC:* 62M10, 42C05, 60G10

## 1. Introduction

Let $\{X_n : n \in \mathbb{Z}\}$ be a real, zero-mean, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$, with spectral measure not of finite support, which we shall simply call a *stationary process*. Here the spectral measure is the finite measure $\mu$ on $(-\pi, \pi]$ in the spectral representation $\gamma(n) = \int_{(\pi,\pi]} e^{in\theta}\mu(d\theta)$ of the *autocovariance function* $\gamma(n) := E[X_n X_0]$, $n \in \mathbb{Z}$. For $\{X_n\}$, we have another sequence $\{\alpha(n)\}_{n=1}^{\infty}$ called the *partial autocorrelation function* (PACF); see (2.1) below for the definition. In the theory of orthogonal polynomials on the unit circle (OPUC), however, the PACF $\{\alpha(n)\}_{n=1}^{\infty}$ appears as the sequence of *Verblunsky coefficients* of the normalized spectral measure $\tilde{\mu} := (\mu(-\pi,\pi])^{-1}\mu$. Notice that $(-\pi,\pi]$ can be identified with the unit circle $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ by the map $\theta \mapsto e^{i\theta}$, whence $\mu$ or $\tilde{\mu}$ with a measure on $\mathbb{T}$. For a survey of OPUC, see Simon (2005a, 2005b, 2005c, 2011).

The Verblunsky coefficients $\{\alpha(n)\}_{n=1}^{\infty}$ give an unrestricted parametrization of the normalized spectral measure $\tilde{\mu}$ of $\{X_n\}$, in that the only inequalities restricting the $\alpha(n)$ are $\alpha(n) \in [-1, 1]$, or $\alpha(n) \in (-1, 1)$ in the non-degenerate case relevant here. This result is due to Barndorff–Nielsen and Schou (1973), Ramsey (1974) in the time-series context. However, in OPUC, the result dates back to Verblunsky (1935, 1936). See, e.g., Simon (2005b, 2005c) and Bingham (2011) for background.

The aim of this paper is to prove an explicit representation of the Verblunsky coefficients $\{\alpha(n)\}_{n=1}^{\infty}$ in terms of another sequence $\{\beta_n\}_{n=0}^{\infty}$ defined by

$$\beta_n := \sum\nolimits_{v=0}^{\infty} c_v a_{v+n}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where $\{c_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ are the MA and AR coefficients of $\{X_n\}$, respectively, defined by (2.3) below. See Inoue and Kasahara (2004 Section 3, 2006 (2.23)). We notice that $\beta_n$ correspond to the Fourier coefficients of the *phase function* of the process (see Remark 1 in §2). The proof of the representation of $\{\alpha(n)\}$ is based on the result of Inoue and Kasahara (2006) on the explicit representation of finite predictor coefficients as well as the Levinson (or Levinson–Durbin) algorithm or the Szegö recursion. The algorithm is due to Szegö (1939), Levinson (1947), and Durbin (1960); for a textbook account, see Pourahmadi (2001, Section 7.2).

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We notice that, in Inoue (2008), the second author already proved a representation of $\{\alpha(n)\}_{n=1}^{\infty}$ in terms of $\{\beta_n\}_{n=0}^{\infty}$. However, the representation of $\{\alpha(n)\}_{n=1}^{\infty}$ in the present paper is much simpler than that in Inoue (2008), in that the latter is of fractional form while the former not. We apply the result to show a general estimate of $\alpha(n)$ for short-memory processes as well as the precise asymptotic behaviour, with remainder, of $\alpha(n)$ for FARIMA processes. The FARIMA model is a popular parametric model with long memory, and was introduced independently by Granger and Joyeux (1980) and Hosking (1981). See Brockwell and Davis (1991, Section 9) for textbook treatment. The long memory of the FARIMA model comes from the singularity at zero of its spectral density.

In §2, we state the main result, i.e., the representation of the Verblunsky coefficients. Its proof is given in §3. In §4, we apply the main result to both short-memory and FARIMA processes.

## 2. Main result

Let $H$ be the real Hilbert space spanned by $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$, which has inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For an interval $I \subset \mathbb{Z}$, we write $H_I$ for the closed subspace of $H$ spanned by $\{X_k : k \in I\}$ and $H_I^{\perp}$ for the orthogonal complement of $H_I$ in $H$. Let $P_I$ and $P_I^{\perp}$ be the orthogonal projection operators of $H$ onto $H_I$ and $H_I^{\perp}$, respectively. The projection $P_I Y$ stands for the best linear predictor of $Y$ based on the observations $\{X_k : k \in I\}$, and $P_I^{\perp} Y$ for its prediction error.

The PACF $\{\alpha(n)\}_{n=1}^{\infty}$ of $\{X_n\}$ is defined by

$$\alpha(1) := \frac{\gamma(1)}{\gamma(0)}, \qquad \alpha(n) := \frac{(P_{[1,n-1]}^{\perp} X_n, P_{[1,n-1]}^{\perp} X_0)}{\|P_{[1,n-1]}^{\perp} X_n\|^2}, \qquad n = 2, 3, \ldots \tag{2.1}$$

(cf. Brockwell and Davis (1991, Sections 3.4 and 5.2)). As stated in §1, The PACF $\{\alpha(n)\}_{n=1}^{\infty}$ coincides with the Verblunsky coefficients of the normalized spectral measure $\tilde{\mu}$. In what follows, we also call $\{\alpha(n)\}_{n=1}^{\infty}$ the Verblunsky coefficients of $\{X_n\}$.

Our main result, i.e., Theorem 2.1 below, is an explicit representation of $\{\alpha(n)\}_{n=1}^{\infty}$. To state it, we need some notation. A stationary process $\{X_n\}$ is said to be *purely nondeterministic* (PND) if $\cap_{n=-\infty}^{\infty} H_{(-\infty,n]} = \{0\}$, or, equivalently, there exists a positive even and integrable function $\Delta$ on $(-\pi, \pi]$ such that $\int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$ and $\gamma_n = \int_{-\pi}^{\pi} e^{in\theta} \Delta(\theta) d\theta$ for $n \in \mathbb{Z}$; see Brockwell and Davis (1991, Section 5.7), Rozanov (1967, Chapter II) and Grenander and Szegö (1958, Chapter 10). We call $\Delta$ the *spectral density* of $\{X_n\}$. Using $\Delta$, we define the *Szegö function* $h$ by

$$h(z) := \sqrt{2\pi} \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta \right\}, \qquad z \in \mathbb{C}, \ |z| < 1. \tag{2.2}$$

The function $h(z)$ is an outer function in the Hardy space $H^2$ of class 2 over the unit disk $|z| < 1$. Using $h$, we define the MA coefficients $c_n$ and the AR coefficients $a_n$, respectively, by

$$h(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad -\frac{1}{h(z)} = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < 1; \tag{2.3}$$

see Inoue (2000, Section 4) and Inoue and Kasahara (2006, Section 2.2) for background. Both $\{c_n\}$ and $\{a_n\}$ are real sequences, and $\{c_n\}$ is in $l^2$.

We write $\mathcal{R}_0$ for the class of *slowly varying functions* at infinity: the class of positive, measurable $\ell$, defined on some neighborhood $[A, \infty)$ of infinity, such that $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$; see Bingham et al. (1989, Chapter 1) for background. Among several possible choices of assumption on $\{X_n\}$, as in Inoue and Kasahara (2006), we consider

**(A1)** $\{X_n\}$ is PND, and both $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} |c_n| < \infty$ hold

as a standard one for processes with short memory, and

**(A2)** $\{X_n\}$ is PND, and, for some $d \in (0, 1/2)$ and $\ell \in \mathcal{R}_0$, $\{c_n\}$ and $\{a_n\}$ satisfy, respectively,

$$c_n \sim n^{-(1-d)} \ell(n), \qquad a_n \sim n^{-(1+d)} \frac{1}{\ell(n)} \cdot \frac{d \sin(\pi d)}{\pi}, \qquad n \to \infty$$

as a standard one for processes with long memory. Here $p_n \sim q_n$ as $n \to \infty$ means $\lim_{n\to\infty} p_n/q_n = 1$.

Recall $\beta_n$ from (1.1). Notice that the sum in (1.1) converges absolutely under either (A1) or (A2). For $n \in \mathbb{N} \cup \{0\}$, we define $\alpha_1(n) := \beta_n$ and, for $k = 3, 5, 7, \ldots$,

$$\alpha_k(n) := \sum\nolimits_{v_1=0}^{\infty} \cdots \sum\nolimits_{v_{k-1}=0}^{\infty} \beta_{n+v_1}\beta_{n+1+v_1+v_2} \cdots \beta_{n+1+v_{k-2}+v_{k-1}}\beta_{n+1+v_{k-1}}.$$

As in the case of $d_k(n, j)$ in Inoue and Kasahara (2006, Section 2.3), the sums converge absolutely. We write $\sum^{\infty-}$ to indicate that the sum does not necessarily converge absolutely, i.e., $\sum_{k=m}^{\infty-} := \lim_{M\to\infty} \sum_{k=m}^{M}$.

Here is the main result of this paper.

**Theorem 2.1.** *We assume either* (A1) *or* (A2)*. Then* $\alpha(n) = \sum_{k=1}^{\infty-} \alpha_{2k-1}(n)$ *for* $n = 2, 3, \ldots$.

The proof of Theorem 2.1 is given in Section 3.

**Remark 1.** We assume $\{a_n\} \in \ell^2$. As usual, we identify $h$ with its boundary-value function $h(e^{i\theta}) = \lim_{r\uparrow 1} h(re^{i\theta})$. Then, since $h(e^{i\theta}) = \sum_{k=0}^{\infty} c_k e^{ik\theta}$ and $1/h(e^{i\theta}) = -\sum_{k=0}^{\infty} e^{ik\theta} a_k e^{ik\theta}$, Parseval's identity yields

$$\int_{-\pi}^{\pi} e^{-in\theta}\{\overline{h(e^{i\theta})}/h(e^{i\theta})\}\frac{d\theta}{2\pi} = -\sum\nolimits_{k=0}^{\infty} c_k a_{k+n} = -\beta_n, \qquad n = 0, 1, \ldots.$$

Here notice that, in our set-up, $\{c_n\}$ is real. Thus $\beta_n$ (or, more precisely, $-\beta_n$) is the $n$-th Fourier coefficient of $\overline{h}/h$. The function $\overline{h}/h$ is called the *phase function* of the process. See Peller (2003, p. 405); see also Dym and McKean (1976) for its continuous-time analogue.

## 3. Proof of Theorem 2.1

In this section, we assume either (A1) or (A2). For $n \in \mathbb{N}$, we can express $P_{[-n,-1]}X_0$ uniquely in the form

$$P_{[-n,-1]}X_0 = \sum\nolimits_{j=1}^{n} \phi_{n,j} X_{-j}.$$

We call $\phi_{n,j}$ the *finite predictor coefficients*. The proof of Theorem 2.1 is based on the explicit representation of $\phi_{n,j}$, i.e., (3.5) below, and the following Szegö recursion (or the Levinson–Durbin algorithm):

$$\phi_{n,j} - \phi_{n+1,j} = \phi_{n,n+1-j}\alpha(n+1), \qquad j = 1, \ldots, n. \tag{3.1}$$

See, e.g., (5.2.4) in Brockwell and Davis (1991) for the latter.

As in Inoue and Kasahara (2006, Section 2.3), we define, for $n, j \in \mathbb{N} \cup \{0\}$,

$$d_0(n, j) = \delta_{j0}, \qquad d_1(n, j) = \beta_{n+j}, \qquad d_2(n, j) = \sum\nolimits_{v_1=0}^{\infty} \beta_{n+j+v_1}\beta_{n+v_1},$$

and

$$d_k(n, j) = \sum\nolimits_{v_1=0}^{\infty} \cdots \sum\nolimits_{v_{k-1}=0}^{\infty} \beta_{n+j+v_{k-1}}\beta_{n+v_{k-1}+v_{k-2}} \cdots \beta_{n+v_2+v_1}\beta_{n+v_1}, \quad k \geq 3,$$

the sums converging absolutely. These satisfy the following recursion: for $n, j \in \mathbb{N} \cup \{0\}$,

$$d_0(n, j) = \delta_{j0}, \qquad d_{k+1}(n, j) = \sum\nolimits_{v=0}^{\infty} \beta_{n+j+v} d_k(n, v), \qquad k \geq 0. \tag{3.2}$$

From the definition of $\alpha_k(n)$ above, we also have

$$\alpha_{2k+1}(n) = \sum\nolimits_{v=0}^{\infty} \beta_{n+v} d_{2k}(n+1, v), \qquad n, k \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

The next proposition is the key to the proof of Theorem 2.1.

**Proposition 3.1.** *For $n, j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, we have*

$$d_{2k}(n, j) - d_{2k}(n+1, j) = \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n) d_{2l-1}(n, j). \tag{3.4}$$

Proof. Let $n, j \in \mathbb{N} \cup \{0\}$. We use mathematical induction on $k$. First, since $\alpha_1(n) = \beta_n$ and $d_1(n, j) = \beta_{n+j}$, we have $d_2(n, j) = \sum_{v=n}^{\infty} \beta_v \beta_{v+j} = \sum_{v=n}^{\infty} \alpha_1(v) d_1(v, j)$, which implies (3.4) with $k = 1$. Next, we assume that (3.4) holds for $k \in \mathbb{N}$. Then, by (3.2) and the Fubini–Tonelli theorem, we have $d_{2k+2}(n, j) = \sum_{v_2=0}^{\infty} [\sum_{v_1=n}^{\infty} \beta_{v_2+v_1} \beta_{j+v_1}] d_{2k}(n, v_2)$, whence $d_{2k+2}(n, j) - d_{2k+2}(n+1, j) = \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I} := \sum\nolimits_{v_2=0}^{\infty} \left[ \sum\nolimits_{v_1=0}^{\infty} \beta_{n+v_2+v_1} \beta_{n+j+v_1} \right] [d_{2k}(n, v_2) - d_{2k}(n+1, v_2)], \qquad \mathrm{II} := \sum\nolimits_{v_2=0}^{\infty} \beta_{n+v_2} \beta_{n+j} d_{2k}(n+1, v_2).$$

By (3.2), (3.4), and the Fubini–Tonelli theorem,

$$\begin{aligned} \mathrm{I} &= \sum\nolimits_{v_2=0}^{\infty} \left[ \sum\nolimits_{v_1=0}^{\infty} \beta_{n+v_2+v_1} \beta_{n+j+v_1} \right] \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n) d_{2l-1}(n, v_2) \\ &= \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n) \sum\nolimits_{v_1=0}^{\infty} \beta_{n+j+v_1} \sum\nolimits_{v_2=0}^{\infty} \beta_{n+v_1+v_2} d_{2l-1}(n, v_2) \\ &= \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n) d_{2l+1}(n, j) = \sum\nolimits_{l=2}^{k+1} \alpha_{2(k+1)-2l+1}(n) d_{2l-1}(n, j), \end{aligned}$$

while, by (3.3), we have $\mathrm{II} = [\sum_{v_2=0}^{\infty} \beta_{n+v_2} d_{2k}(n+1, v_2)] \beta_{n+j} = \alpha_{2k+1}(n) d_1(n, j)$. Thus we obtain (3.4) with $k$ replaced by $k+1$, as desired. □

For $n \in \mathbb{N}$ and $j = 1, \ldots, n$, Theorem 2.9 in Inoue and Kasahara (2006) asserts the representation

$$\phi_{n,j} = \sum\nolimits_{k=1}^{\infty-} \{b_{2k-1}(n, j) + b_{2k}(n, n+1-j)\}, \tag{3.5}$$

where

$$b_k(n, j) = c_0 \sum\nolimits_{u=0}^{\infty} a_{j+u} d_{k-1}(n+1, u), \qquad k = 1, 2, \ldots. \tag{3.6}$$

Using Proposition 3.1, we derive two kinds of difference equations for $b_k(n, j)$.

**Proposition 3.2.** *For $n, k \in \mathbb{N}$ and $j = 1, \ldots, n$, we have*

$$b_{2k+1}(n, j) - b_{2k+1}(n+1, j) = \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n+1) b_{2l}(n, j), \tag{3.7}$$

$$b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j) = \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n+1) b_{2l-1}(n, n+1-j). \tag{3.8}$$

Proof. From (3.4) and (3.6), we easily obtain (3.7) in the following way:

$$\begin{aligned} b_{2k+1}(n, j) - b_{2k+1}(n+1, j) &= c_0 \sum\nolimits_{u=0}^{\infty} a_{j+u} \{d_{2k}(n+1, u) - d_{2k}(n+2, u)\} \\ &= c_0 \sum\nolimits_{u=0}^{\infty} a_{j+u} \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n+1) d_{2l-1}(n+1, u) \\ &= \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n+1) \left( c_0 \sum\nolimits_{u=0}^{\infty} a_{j+u} d_{2l-1}(n+1, u) \right) = \sum\nolimits_{l=1}^{k} \alpha_{2k-2l+1}(n+1) b_{2l}(n, j). \end{aligned}$$

We turn to (3.8). Since $d_1(n+1, u) = \alpha_1(n+1+u)$ and $b_1(n, j) = c_0 a_j$, it follows from (3.6) that

$$b_2(n, n+1-j) = c_0 \sum\nolimits_{u=0}^{\infty} a_{n+1-j+u} d_1(n+1, u) = \sum\nolimits_{u=n+1}^{\infty} \alpha_1(u) b_1(n, u-j).$$

Similarly, $b_2(n+1, n+2-j) = c_0 \sum_{u=0}^{\infty} a_{n+2-j+u} d_1(n+2, u) = \sum_{u=n+2}^{\infty} \alpha_1(u) b_1(n, u-j)$. Thus (3.8) holds for $k = 1$. If $k \geq 2$, then, by (3.2) and (3.6), we have

$$\begin{aligned} b_{2k}(n, n+1-j) &= c_0 \sum\nolimits_{u=0}^{\infty} a_{n+1-j+u} \sum\nolimits_{v=0}^{\infty} \beta_{n+1+u+v} d_{2(k-1)}(n+1, v), \\ b_{2k}(n+1, n+2-j) &= c_0 \sum\nolimits_{u=1}^{\infty} a_{n+1-j+u} \sum\nolimits_{v=0}^{\infty} \beta_{n+1+u+v} d_{2(k-1)}(n+2, v), \end{aligned}$$

whence $b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j) = \mathrm{I} + \mathrm{II}$ with

$$\begin{aligned}\mathrm{I} &:= c_0 \sum_{u=0}^{\infty} a_{n+1-j+u} \sum_{v=0}^{\infty} \beta_{n+1+u+v} \left\{d_{2(k-1)}(n+1, v) - d_{2(k-1)}(n+2, v)\right\}, \\ \mathrm{II} &:= c_0 a_{n+1-j} \sum_{v=0}^{\infty} \beta_{n+1+v} d_{2(k-1)}(n+2, v).\end{aligned}$$

By (3.2), (3.4) and (3.6),

$$\begin{aligned}\mathrm{I} &= c_0 \sum_{u=0}^{\infty} a_{n+1-j+u} \sum_{v=0}^{\infty} \beta_{n+1+u+v} \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) d_{2l-1}(n+1, v) \\ &= \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) \left(c_0 \sum_{u=0}^{\infty} a_{n+1-j+u} \sum_{u=0}^{\infty} \beta_{n+1+u+v} d_{2l-1}(n+1, v)\right) \\ &= \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) b_{2l+1}(n, n+1-j) = \sum_{l=2}^{k} \alpha_{2k-2l+1}(n+1) b_{2l-1}(n, n+1-j),\end{aligned}$$

while, by (3.3) and $b_1(n, n+1-j) = ca_{n+1-j}$, we have $\mathrm{II} = \alpha_{2k-1}(n+1) b_1(n, n+1-j)$. Thus (3.8) follows. □

We are now ready to prove Theorem 2.1.

Proof (of Theorem 2.1). For $n \in \mathbb{N}$ and $j = 1, \ldots, n$, we have $b_1(n, j) - b_1(n+1, j) = c_0 a_j - c_0 a_j = 0$. This, together with (3.5), (3.7) and (3.8), yields

$$\begin{aligned}\phi_{n,j} - \phi_{n+1,j} &= \sum_{k=1}^{\infty-} \{b_{2k+1}(n, j) - b_{2k+1}(n+1, j) + b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j)\} \\ &= \sum_{k=1}^{\infty-} \sum_{l=1}^{k} \alpha_{2k-2l+1}(n+1) \{b_{2l}(n, j) + b_{2l-1}(n, n+1-j)\} \\ &= \sum_{l=1}^{\infty-} \{b_{2l}(n, j) + b_{2l-1}(n, n+1-j)\} \sum_{k=l}^{\infty-} \alpha_{2k-2l+1}(n+1) = \left\{\sum_{k=1}^{\infty-} \alpha_{2k-1}(n+1)\right\} \phi_{n,n+1-j}.\end{aligned}$$

Since $P_{[-n,-1]} X_0 \neq 0$, we have $(\phi_{n,1}, \ldots, \phi_{n,n}) \neq 0$. Combining these and (3.1), we obtain the theorem. □

## 4. Applications

### 4.1. *Short memory processes*

In this subsection, we apply Theorem 2.1 to the Verblunsky coefficients of short-memory processes.

We define

$$F(j) := \left\{\sum_{v=0}^{\infty} |c_v|\right\} \left\{\sum_{u=j}^{\infty} |a_u|\right\}, \qquad j = 0, 1, \ldots.$$

Then $F(j)$ decreases to zero as $j \to \infty$ under (A1). Recall $d_k(n, j)$ from Section 3.

**Lemma 4.1.** *We assume* (A1). *Then* $\sum_{u=0}^{\infty} |d_k(n, u)| \leq F(n)^k$ *for* $k, n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. We use induction on $k$. Since $d_1(n, u) = \beta_{n+u}$, we have

$$\sum_{u=0}^{\infty} |d_1(n, u)| = \sum_{u=0}^{\infty} |\beta_{n+u}| \leq \sum_{v=0}^{\infty} |c_v| \sum_{u=0}^{\infty} |a_{n+u+v}| \leq F(n).$$

We assume $\sum_{u=0}^{\infty} |d_k(n, u)| \leq F(n)^k$ for $k \in \mathbb{N}$. Then, by (3.2),

$$\sum_{u=0}^{\infty} |d_{k+1}(n, u)| \leq \sum_{v=0}^{\infty} |d_k(n, v)| \sum_{u=0}^{\infty} |\beta_{n+v+u}| \leq F(n) \sum_{v=0}^{\infty} |d_k(n, v)| \leq F(n)^{k+1}.$$

Thus the inequality also holds for $k+1$. □

Notice that $\{a_n\} \in \ell^1$ implies $a_n \to 0$ as $n \to \infty$.

**Theorem 4.2.** *We assume* (A1). *Then, for* $N \in \mathbb{N}$ *such that* $F(N+1) < 1$, *the Verblunsky coefficients* $\{\alpha(n)\}$ *satisfy*

$$|\alpha(n)| \leq \frac{\sum_{v=0}^{\infty} |c_v|}{1 - F(n+1)^2} \left(\max_{j \geq n} |a_j|\right), \qquad n \geq N.$$

Proof. Recall $\alpha_{2k+1}(n)$ from §2. We have $|\alpha_1(n)| = |\beta_n| \le \left(\max_{j\ge n}|a_j|\right)\sum_{v=0}^{\infty}|c_v|$. By (3.3) and Lemma 4.1, we also have $|\alpha_{2k+1}(n)| \le \sum_{v=0}^{\infty}|\beta_{n+v}d_{2k}(n+1,v)| \le F(n+1)^{2k}\left(\max_{j\ge n}|a_j|\right)\sum_{v=0}^{\infty}|c_v|$ for $n,k\in\mathbb{N}$. Choose $N\in\mathbb{N}$ so that $F(N+1)<1$. Then, combining the estimates above with Theorem 2.1, we see that, for $n\ge N$,

$$|\alpha(n)| \le \left(\max_{j\ge n}|a_j|\right)\sum_{v=0}^{\infty}|c_v|\sum_{k=0}^{\infty}F(n+1)^{2k} = \frac{\sum_{v=0}^{\infty}|c_v|}{1-F(n+1)^2}\left(\max_{j\ge n}|a_j|\right).$$

Thus the theorem follows. □

For example, if, in addition to (A1), $a_n = O(n^{-p})$ as $n\to\infty$ for some $p>1$, then $\max_{j\ge n}|a_j| = O(n^{-p})$, whence, by Theorem 4.2, $\alpha(n) = O(n^{-p})$ as $n\to\infty$.

*4.2. The FARIMA model*

For $d\in(-1/2,1/2)$ and $p,q\in\mathbb{N}\cup\{0\}$, a stationary process $\{X_n\}$ is said to be a FARIMA$(p,d,q)$ (or fractional ARIMA$(p,d,q)$) process if it has a spectral density $\Delta$ of the form

$$\Delta(\theta) = \frac{1}{2\pi}\frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2}|1-e^{i\theta}|^{-2d}, \qquad -\pi<\theta\le\pi,$$

where $\Phi(z)$ and $\Theta(z)$ are polynomials with real coefficients of degrees $p$, $q$, respectively, satisfying the following condition: $\Phi(z)$ and $\Theta(z)$ have no common zeros, and have no zeros in the closed unit disk $\{z\in\mathbb{C} \mid |z|\le 1\}$.

In what follows, we assume that $\{X_n\}$ is a FARIMA$(p,d,q)$ process with $0<d<1/2$. Then $\{X_n\}$ satisfies (A2) for some constant function $\ell$ (cf. Inoue (2002, Corollary 3.1)). Let $\{\alpha(n)\}$ be the Verblunsky coefficients of $\{X_n\}$. The aim of this subsection is to apply Theorem 2.1 to $\{\alpha(n)\}$ to prove the next theorem.

**Theorem 4.3.** *We have $n\alpha(n) = d + O(n^{-d})$ as $n\to\infty$.*

Theorem 4.3 is more precise than Inoue (2008, Theorem 2.5) with $0<d<1/2$, in that the former gives an estimate on the remainder term. The rest of this subsection is devoted to the proof of Theorem 4.3.

As before, we denote by $\{c_n\}$ and $\{a_n\}$ the MA and AR coefficients, respectively, of $\{X_n\}$. We also consider a FARIMA$(0,d,0)$ process $\{X'_n\}$ satisfying $E[(X'_n)^2] = \Gamma(1-2d)/\Gamma(1-d)^2$. The AR coefficients $\{a'_n\}$ and MA coefficients $\{c'_n\}$ of $\{X'_n\}$ are given by

$$a'_n = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}, \qquad c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)}, \qquad n=0,1,\ldots$$

(see, e.g., Brockwell and Davis (1991, Section 13.2)). Notice that $c'_n>0$ for $n\ge 0$ and $a'_n>0$ for $n\ge 1$. Put

$$\beta'_n := \sum_{v=0}^{\infty}c'_v a'_{n+v}, \qquad n=0,1,\ldots.$$

**Lemma 4.4.** *We have $\beta'_n = \sin(\pi d)/\{\pi(n-d)\}$ for $n=0,1,\ldots$.*

Proof. Using the hypergeometric function, we have, for $n\ge 0$,

$$\begin{aligned}\beta'_n &= \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}\sum_{v=0}^{\infty}\frac{\Gamma(d+v)}{\Gamma(d)}\cdot\frac{\Gamma(n-d+v)}{\Gamma(n-d)}\cdot\frac{\Gamma(n+1)}{\Gamma(n+1+v)}\cdot\frac{1}{v!} = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}F(d,n-d;n+1;1)\\ &= \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}\cdot\frac{\Gamma(n+1)}{\Gamma(n+1-d)\Gamma(1+d)} = \frac{\sin(\pi d)}{\pi}\cdot\frac{1}{n-d},\end{aligned}$$

as desired. □

**Proposition 4.5.** *There exist a real sequence $\{\delta_n\}_{n=1}^{\infty}$ and $M\in(0,\infty)$ such that $\beta_n = \beta'_n\{1+\delta_n\}$ and $|\delta_n|\le Mn^{-d}$ for $n\in\mathbb{N}$.*

Proof. By Inoue (2002, Lemma 2.2), we have, as $n \to \infty$,

$$\frac{c_n}{n^{d-1}} = \frac{K_1}{\Gamma(d)} + O\left(n^{-1}\right), \quad \frac{a_n}{n^{-d-1}} = -\frac{1}{K_1\Gamma(-d)} + O\left(n^{-1}\right), \quad \frac{c'_n}{n^{d-1}} = \frac{1}{\Gamma(d)} + O\left(n^{-1}\right), \quad \frac{a'_n}{n^{d-1}} = -\frac{1}{\Gamma(-d)} + O\left(n^{-1}\right),$$

where $K_1 := \theta(1)/\phi(1) > 0$. Hence we may write $c_n = \{K_1 + s_n\}c'_n$ for $n \geq 0$ and $a_n = \{(1/K_1) + t_n\}a'_n$ for $n \geq 1$, where $\{s_n\}$ and $\{t_n\}$ are sequences satisfying $|s_n| \leq L/(n+1)$ for $n \geq 0$ and $|t_n| \leq L/n$ for $n \geq 1$, for some $L \in (0, \infty)$.

We have, for $n = 1, 2, \ldots$,

$$|\beta_n - \beta'_n| \leq \sum_{v=0}^{\infty} |s_v| c'_v a'_{n+v} + K_1 \sum_{v=0}^{\infty} |t_{n+v}| c'_v a'_{n+v} + (1/K_1) \sum_{v=0}^{\infty} |s_v t_{n+v}| c'_v a'_{n+v}.$$

From $c'_n/(n+1) \sim 1/\{n^{2-d}\Gamma(d)\}$ as $n \to \infty$, we see that

$$\sum_{v=0}^{\infty} |s_v| c'_v a'_{n+v} \leq L \sum_{v=0}^{\infty} \frac{c'_v}{v+1} a'_{n+v} \sim a'_n L \sum_{v=0}^{\infty} \frac{c'_v}{v+1}, \qquad n \to \infty.$$

Hence, using $a'_n \sim \text{constant} \cdot n^{-(1+d)}$ as $n \to \infty$, we get $\sum_{v=0}^{\infty} |s_v| c'_v a'_{n+v} = O\left(n^{-(1+d)}\right)$ as $n \to \infty$. Similarly, as $n \to \infty$,

$$\sum_{v=0}^{\infty} |t_{n+v}| c'_v a'_{n+v} = O\left(n^{-(2+d)}\right), \qquad \sum_{v=0}^{\infty} |s_v t_{n+v}| c'_v a'_{n+v} = O\left(n^{-(2+d)}\right).$$

Combining these and $\beta'_n \sim \pi^{-1} \sin(\pi d)\, n^{-1}$, we obtain the proposition. □

Proof (of Theorem 4.3). By Theorem 2.1, the Verblunsky coefficients $\{\alpha(n)\}$ of $\{X_n\}$ and $\{\alpha'(n)\}$ of $\{X'_n\}$ admit the representations $\alpha(n) = \sum_{k=1}^{\infty-} \alpha_{2k-1}(n)$ and $\alpha'(n) = \sum_{k=1}^{\infty-} \alpha'_{2k-1}(n)$, respectively, where $\alpha_{2k-1}(\cdot)$ are those defined for $\{X_n\}$ in §2, while $\alpha'_{2k-1}(\cdot)$ are their counterparts defined for $\{X'_n\}$, that is, $\alpha'_1(n) = \beta'_n$ and, for $k = 3, 5, \ldots$,

$$\alpha'_k(n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_{k-1}=0}^{\infty} \beta'_{n+1+m_1} \beta'_{n+1+m_1+m_2} \cdots \beta'_{n+1+m_{k-2}+m_{k-1}} \beta'_{n+m_{k-1}}.$$

For $k \in \mathbb{N}$, we define $\tau_{2k-1} \in (0, \infty)$ by $\tau_{2k-1} := (2k-2)!/[\pi 2^{2k-2}((k-1)!)^2(2k-1)]$, or by

$$\sum_{k=1}^{\infty} \tau_{2k-1} x^{2k-1} = \pi^{-1} \arcsin x, \qquad |x| < 1 \tag{4.1}$$

(see Inoue and Kasahara (2006, Lemma 3.1) and Inoue (2008, Section 5)). Let $M$ be as in Proposition 4.5 and let $r > 1$ be chosen so that $r^2 \sin(\pi d) < 1$. Then, as in the proof of Inoue and Kasahara (2006, Proposition 3.2), there exists an integer $N$ independent of $k$ such that

$$1 + (M/n^d) \leq r, \qquad \alpha'_{2k-1}(n) \leq \frac{1}{n}\{r \sin(\pi d)\}^{2k-1} \tau_{2k-1}, \qquad n \geq N,\ k \geq 1.$$

By Proposition 4.5, we have $|\beta_{n+v}| \leq \left(1 + Mn^{-d}\right)\beta'_{n+v}$ and $|\beta_{n+v} - \beta'_{n+v}| \leq Mn^{-d}\beta'_{n+v}$ for $n \geq 1$ and $v \geq 0$. We also have $(1+x)^k - 1 \leq kx(1+x)^k$ for $x \geq 0$. Hence, for $n \geq N$,

$$\begin{aligned}
&|\alpha_3(n) - \alpha'_3(n)| \leq \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} |\beta_{n+1+m_1} - \beta'_{n+1+m_1}| \cdot |\beta_{n+1+m_1+m_2}| \cdot |\beta_{n+m_2}| \\
&+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \beta'_{n+1+m_1} |\beta_{n+1+m_1+m_2} - \beta'_{n+1+m_1+m_2}| \cdot |\beta_{n+m_2}| + \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \beta'_{n+1+m_1} \beta'_{n+1+m_1+m_2} |\beta_{n+m_2} - \beta'_{n+m_2}| \\
&\leq Mn^{-d}\left\{(1 + Mn^{-d})^2 + (1 + Mn^{-d}) + 1\right\}\alpha'_3(n) = \left\{(1 + Mn^{-d})^3 - 1\right\}\alpha'_3(n) \\
&\leq 3Mn^{-d}(1 + Mn^{-d})^3 \alpha'_3(n) \leq 3Mn^{-(d+1)}\{r^2 \sin(\pi d)\}^3 \tau_3.
\end{aligned}$$

In the same way, $|\alpha_{2k-1}(n) - \alpha'_{2k-1}(n)| \leq (2k-1)Mn^{-(d+1)}\{r^2 \sin(\pi d)\}^{2k-1}\tau_{2k-1}$ for $k = 1, 2, \ldots$ and $n \geq N$.

Since $\alpha'(n) = d/(n-d)$ (see Hosking (1981, Theorem 1)), it follows that

$$\left|\alpha(n) - \frac{d}{n-d}\right| \leq \sum_{k=1}^{\infty} |\alpha_{2k-1}(n) - \alpha'_{2k-1}(n)| \leq n^{-(d+1)} M \sum_{k=1}^{\infty} (2k-1)\tau_{2k-1}\{r^2 \sin(\pi d)\}^{2k-1}.$$

By (4.1), we have $\sum_{k=1}^{\infty} (2k-1)\tau_{2k-1}\{r^2 \sin(\pi d)\}^{2k-1} < \infty$, so that

$$\alpha(n) = \frac{d}{n-d} + O\left(n^{-(d+1)}\right) = \frac{d}{n} + O\left(n^{-(d+1)}\right), \qquad n \to \infty.$$

Thus the theorem follows. □

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